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Curves of Marginal Stability, and Weak and Strong-Coupling BPS Spectra in $N = 2$ Supersymmetric QCD

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ABSTRACT

We explicitly determine the global structure of the $SL(2, \mathbf{Z})$ bundle over the Coulomb branch of the moduli space of asymptotically free $N = 2$ supersymmetric Yang-Mills theories with gauge group $SU(2)$ when massless hypermultiplets are present. For each relevant number of flavours, we show that there is a curve of marginal stability on the Coulomb branch, diffeomorphic to a circle, across which the BPS spectrum is discontinuous. We determine rigorously and completely the BPS spectra inside and outside the curve. In all cases, the spectrum inside the curve consists of only those BPS states that are responsible for the singularities of the low energy effective action (in addition to the massless abelian gauge multiplet which is always present). The predicted decay patterns across the curve of marginal stability are perfectly consistent with all quantum numbers carried by the BPS states. As a byproduct, we also show that the electric and magnetic quantum numbers of the massless states at the singularities proposed by Seiberg and Witten are the only possible ones.

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1. Introduction and Summary

Recently, some new methods have been imagined which led to the complete determination of the spectrum of BPS states[‡] in $N = 2$ supersymmetric theories (first in [1] and later in [2]). Part of these results had been anticipated at the conjectural level [3,4]. The problem was open since the work of Seiberg and Witten [5,6] where it was realized that there may exist some regions in the moduli space where the BPS spectrum cannot be continuously related to the semi-classical one. This stems from the fact that the duality transformations (monodromies) involved in the description of the low energy physics of these theories are not quantum symmetries at all energy scales. In particular, the semi-classical BPS spectrum need not to be invariant under the full monodromy group, but only under a subgroup of it, generated by the monodromy at infinity M_∞, which shifts the electric charge of a dyon by an integer multiple of its magnetic charge. The discontinuity in the BPS spectrum is possible because there exists a curve of marginal stability where usually stable BPS particles become degenerate in mass with other BPS states. This phenomena was first highlighted in two dimensional theories in [7]. As for the pure $N_f = 0$ gauge theory, we will show here that in the cases $N_f = 1, 2, 3$ these curves separate what we call a strong-coupling from a weak-coupling region. The latter name is used for the region that contains the semi-classical one, where the Higgs vacuum expectation value goes to infinity and the theories are weakly coupled. On the other hand, this so-called weak-coupling region extends all the way to the curve of marginal stability where the physics is actually strongly coupled. The existence of this curve is a genuine non-perturbative effect in the theories with zero bare masses.[§]

In [1] the authors have shown that in the $N_f = 0$ theory the strong-coupling BPS spectrum must be invariant under some duality transformations which *do not* belong to the monodromy group, but which are related to the existence of a global exact quantum symmetry acting on the moduli space. There it was a $\mathbf{Z}_2$ symmetry coming from the spontaneous breaking of a $\mathbf{Z}_8$ global symmetry down to $\mathbf{Z}_4$. Surprisingly, combining this symmetry with the fact that the low energy physics cannot be described using a unique set of elementary light fields over the whole moduli space (that is, the $SL(2, \mathbf{Z})$ bundle is not trivial), it was shown in [1] that, in the

‡ By BPS spectrum, we always mean the set of BPS quantum states existing in the theory, not the mass spectrum.

§ Note, however, that for large bare masses of the quarks, such curves also exist at weak coupling where jumps in the spectrum can be studied using semi-classical methods [8].

strong-coupling region, the BPS states must come in *multiplets of the broken symmetry* (the "$\mathbf{Z}_2$ pairs"). This was at the basis of the proof that the strong-coupling spectrum consists only of the particles responsible for the singularities, namely the magnetic monopole $(n_e, n_m) = \pm(0, 1)$ and the dyon of unit electric charge (in the normalisations of [1,5]), alternatively described as $\pm(\pm 1, 1)$. Also included in [1] was a new way of determining the weak-coupling spectrum, independent of the semi-classical approach. It was shown that no state with a magnetic charge $|n_m| \geq 2$ can be present. As noted by Sen [9] this can also be viewed as a consequence of the conjectured self-duality of the $N = 4$ theory. The same result also follows solely from semi-classical reasoning as noted in [10]. The results of [1] have been subsequently confirmed in [2] where a completely independent method from string theory was used.

In the present paper, we will extend and generalize the methods and results of [1] to the case where $N_f = 1$, 2 or 3 hypermultiplets (quarks), without bare masses, in the spin 1/2 representation of the gauge group $SU(2)$, are included. If not for the bare masses set to zero, these are the most general asymptotically free theories of this type. For $N_f = 1$, the moduli space of vacua $\mathcal{M}_1$ is homeomorphic to the compactified complex u-plane, where $u = \langle \mathrm{tr}\, \phi^2 \rangle$ and ϕ is the Higgs field in the adjoint representation of $SU(2)$. This is a Coulomb branch on which the gauge group is broken down to $U(1)$. For $N_f \geq 2$, in addition to a similar Coulomb branch, the moduli space has also Higgs branches on which the gauge group is completely broken. In this paper we focus entirely on the Coulomb branch where magnetically charged particles exist. We will denote this Coulomb branch as $\mathcal{M}_{N_f}$ in the following. The expectation value $a(u)$ of the scalar field ϕ and its dual $a_D(u)$ are given by the two periods of certain one-forms on appropriate elliptic curves [6]. As such they satisfy Picard-Fuchs differential equations. These Picard-Fuchs equations were derived in [11] where solutions, valid in the neighbourhoods of the various singularities, were also given. However, though this was enough to determine power series expansions for the prepotential and its dual, no explicit expressions valid on the whole Coulomb branch were given. In the following, we will express a_D and a in terms of standard hypergeometric functions and explicitly display the global analytic structure of the $SL(2,\mathbf{Z})$ bundle E_{N_f} over $\mathcal{M}_{N_f}$. This will prove useful when discussing the strong-coupling spectrum.

Once we have these solutions, it is indeed straightforward to determine the curves of marginal stability where a_D/a is real. For each $N_f = 1$, 2, 3, one has a single closed curve

diffeomorphic to a circle. Of course, since a BPS state can become massless only at points on this curve, as discussed in [1], the curve goes through all singular points of the Coulomb branch (except $u = \infty$). For $N_f = 2$, the solutions $a_D(u)$, $a(u)$, up to a multiplicative constant, and hence the curve are identical to those of $N_f = 0$. For $N_f = 3$, the curve is a rescaled and shifted version of the $N_f = 0$ curve.

To determine the spectra, we will in particular exploit the global $\mathbf{Z}_{4-N_f}$ symmetry acting on the Coulomb branch of the moduli space. From this point of view, the richest structure is for $N_f = 1$ which exhibits a $\mathbf{Z}_3$ symmetry. For $N_f = 3$ there is no such symmetry, but we will show that there it is not needed for the determination of the spectrum. In this case, the strong-coupling spectrum consists of only the dyon* $(-1, 2)$ with magnetic charge two and the magnetic monopole $(0, 1)$. The weak-coupling spectrum contains the dyons $(n, 1)$ and $(2n + 1, 2)$ for all integers n, in addition to the elementary quarks $(1, 0)$ and W bosons $(2, 0)$, while we prove that no magnetic charges $|n_m| \geq 3$ can exist. For $N_f = 2$, we have only $(0, 1)$ and $(1, 1)$ in the strong-coupling region, the semi-classical spectrum consisting of all the dyons $(n, 1)$ where n is any integer, together with the quarks and W bosons. We show that no $|n_m| \geq 2$ are allowed. For $N_f = 1$, the strong-coupling spectrum consists again of only the three states responsible for the singularities, namely the monopole $(0, 1)$ and the dyons $(-1, 1)$ and $(-2, 1)$. For the weak-coupling spectrum we prove that again no magnetic charges $|n_m| \geq 2$ can exist, while it contains all dyons $(n, 1)$, $n \in \mathbf{Z}$, as well as the elementary quarks and W bosons. Finally note that the neutral abelian $N = 2$ vector multiplet is present in all cases over the whole moduli space.

We begin in Section 2 with a brief overview of the results obtained in [1] for $N_f = 0$ and then discuss some important facts and ideas we will use in this work. There, we also discuss in some detail the rôle of the broken global symmetries and the relevant representations of the flavour groups. Then, in Sections 3 to 5, we present our results for respectively $N_f = 1$, 2 and 3. These sections are organized as follows: we start by presenting the structure of the singularities on the Coulomb branch, and explain why it is unique. The global analytic structure of the $SL(2,\mathbf{Z})$ bundle E_{N_f} is then determined, and the curve $\mathcal{C}_{N_f}$ of marginal stability described. When it exists, we also study the global quantum symmetry $\mathbf{Z}_{4-N_f}$ acting on $\mathcal{M}_{N_f}$. This leads

$\star$ Note that we changed the sign of the electric charge n_e with respect to the conventions of Seiberg and Witten [5,6].

to the determination of the BPS spectra. We then show that the predicted decay reactions across the curve $\mathcal{C}_{N_f}$ are perfectly consistent with all quantum numbers carried by the BPS states, and in particular with the branching rules of the representations of the flavour groups.

2. Review of $N_f = 0$, and general considerations

In this Section, we review or introduce certain results and ideas, which will prove very useful in the following. Along the Coulomb branch, the effective action is given in terms of a prepotential $\mathcal{F}(a)$, and the knowledge of $a(u)$ and $a_D(u) \equiv \frac{1}{2}\frac{\partial \mathcal{F}}{\partial a}(a(u))$ on all of the Coulomb branch $\mathcal{M}$, which is the (compactified) u-plane, allows one to reconstruct $\mathcal{F}(a)$. Also the mass of any BPS state with electric and magnetic charges n_e and n_m is determined as[†]

$$m = \sqrt{2}\left|a n_e - a_D n_m\right| = \sqrt{2}\left|\eta\big((n_e, n_m), (a_D, a)\big)\right| \,, \tag{2.1}$$

where η is the standard symplectic product. In Seiberg and Witten's work [5,6], the physical requirement of having certain monodromies around singularities in the moduli space is translated by equating da_D/du and da/du with period integrals of the only holomorphic one-form on an appropriate genus one Riemann surface. The monodromies are determined by the asymptotic behaviour of (a_D, a) near the relevant singularities. Near the point at infinity the one loop β function of the microscopic theory leads to

$$\begin{aligned} a^{(N_f)}(u) &\sim \frac{1}{2}\sqrt{2u} \\ a_D^{(N_f)}(u) &\sim \frac{i}{4\pi}\,(4 - N_f)\,\sqrt{2u}\,\log u \end{aligned} \tag{2.2}$$

and

$$M_\infty = \begin{pmatrix} -1 & 4 - N_f \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

The singularities at finite u are due to BPS states (n_e, n_m) becoming massless. Using the β function of the abelian low energy theory in which (n_e, n_m) is coupled locally, one can obtain the corresponding monodromy matrix. When d BPS hypermultiplets (n_e, n_m) become

† We will equally note the sections of E_{N_f} by line or column vectors.

massless, which occurs when the states carry a d dimensional representation of the flavour group, we have

$$M_{(n_e,n_m),d} = \begin{pmatrix} 1 - n_e n_m\, d & n_e^2\, d \\ -n_m^2\, d & 1 + n_e n_m\, d \end{pmatrix}. \tag{2.4}$$

The easiest way of determining the explicit solution for (a_D, a) is to use the Picard-Fuchs equation satisfied by the period integrals. These have been derived in [11] for the elliptic curves given in [6], and are second order differential equations having two linearly independent solutions. The knowledge of the asymptotics fixes the appropriate linear combinations for a_D and a. This will be worked out for $N_f = 1$ and $N_f = 3$ below.

We also recall that it follows from (2.1) that for given u the mass of any BPS state is proportional to the euclidean length of the vector $n_e a - n_m a_D$ that lies on the lattice spanned by the two complex numbers $a(u)$ and $a_D(u)$. Charge conservation and the triangle inequality then imply that a state (n_e, n_m) can only decay if n_e and n_m are not relatively prime, i.e. if $(n_e, n_m) = q(n, m)$ with $n, m, q \in \mathbf{Z}$, $q \neq \pm 1$. Hence, BPS states with n_e, n_m relatively prime are stable. This argument fails if $a_D/a \in \mathbf{R}$, so that the lattice collapses onto a single line. On the moduli space $\mathcal{M}_{N_f}$, the curve of marginal stability $\mathcal{C}_{N_f}$ is defined as $\mathcal{C}_{N_f} = \Big\{ u \in \mathcal{M} \;\big|\; a_D^{(N_f)}/a^{(N_f)} \in \mathbf{R} \Big\}$. Hence, at any two points u and u' that can be joined by a path that does not cross the curve $\mathcal{C}$ the spectrum of stable BPS states is necessarily the same. As already mentioned in the introduction we call the BPS spectra outside and inside the curve the weak and strong-coupling spectra, denoted $\mathcal{S}_W$ and $\mathcal{S}_S$, and refer to the regions as $\mathcal{R}_W$ and $\mathcal{R}_S$.

2.1. Review of $N_f = 0$

For a pedagogical introduction to this case, see e.g. [12]. We will use the normalization convention of [6] for the electric charge, so that n_e is always an integer. Indeed, the Dirac quantization condition allows states having half the electric charge of the W bosons which now have $n_e = 2$. Though this possibility is not realized in the pure gauge theory, for $N_f > 0$ it will be, the elementary quarks $(\pm 1, 0)$ giving an example. Also, a is now defined by $\langle \phi \rangle = a \sigma^3$. Thus with respect to the "old" conventions of [5,1] we multiply n_e by 2 and divide a by 2.

Then (2.1) is still the correct mass formula. The Picard-Fuchs equation is

$$\left[(u^2-1)\frac{\mathrm{d}^2}{\mathrm{d}u^2}+\frac{1}{4}\right]\begin{pmatrix} a_D \\ a \end{pmatrix}(u)=0\ , \tag{2.5}$$

which is equivalent to a hypergeometric differential equation with parameters $a=b=-\frac{1}{2}$, $c=0$. These parameters are not generic but imply that the solutions are of the logarithmic type [13] which in turn ensures that one gets $SL(2,\mathbf{Z})$ valued monodromy matrices. The requirement that $a_D(u)$ vanishes at $u=1$, so that, according to (2.1), the magnetic monopole with $(n_e,n_m)=(0,1)$ has vanishing mass at this point, dictates the choice for $a_D(u)$. Similarly, the asymptotics $a(u)\sim\sqrt{u/2}$ as $u\to\infty$ determines the choice for $a(u)$. One has

$$\begin{aligned} a_D(u) &= i\,\frac{u-1}{2}\,F\left(\frac{1}{2},\frac{1}{2},2;\frac{1-u}{2}\right) \\ a(u) &= \left(\frac{u+1}{2}\right)^{\frac{1}{2}}F\left(-\frac{1}{2},\frac{1}{2},1;\frac{2}{u+1}\right)\ , \end{aligned} \tag{2.6}$$

where F is the standard hypergeometric function, see e.g. [13]. Here and in the following, the argument of a complex number always runs from $-\pi$ to π. Thus, $a_D(u)$ has a cut on the real line from $-\infty$ to -1, while $a(u)$ has two cuts, both on the real line, one from $-\infty$ to -1 and another from -1 to 1, see Fig. 1.

Fig. 1: The branch cuts of $a_D(u)$ (left) and of $a(u)$ (right).

The asymptotic behaviour of the functions $a_D(u)$ and $a(u)$ as $u\to\infty,1,-1$ is [1]:

$$\left.\begin{aligned} a_D(u) &\simeq \frac{i}{\pi}\sqrt{2u}\left[\log u+3\log 2-2\right] \\ a(u) &\simeq \frac{1}{2}\sqrt{2u} \end{aligned}\right\}\quad \text{as } u\to\infty$$

$$\left.\begin{aligned} a_D(u) &\simeq i\,\frac{u-1}{2} \\ a(u) &\simeq \frac{2}{\pi}-\frac{1}{2\pi}\frac{u-1}{2}\log\frac{u-1}{2}+\frac{1}{2\pi}\frac{u-1}{2}(-1+4\log 2) \end{aligned}\right\}\quad \text{as } u\to 1$$

$$\left.\begin{aligned} a_D(u) &\simeq \frac{i}{\pi}\left[-\frac{u+1}{2}\log\frac{u+1}{2}+\frac{u+1}{2}(1+4\log 2)-4\right] \\ a(u) &\simeq \frac{i}{2\pi}\left[\epsilon\frac{u+1}{2}\log\frac{u+1}{2}+\frac{u+1}{2}\Big(-i\pi-\epsilon\big(1+4\log 2\big)\Big)+4\epsilon\right] \end{aligned}\right\} \quad \text{as } u\to -1\ , \tag{2.7}$$

where ϵ is the sign of $\Im m\, u$. The monodromy matrices around the singular points $u = -1, 1, \infty$ can then be read off from the different asymptotics. One has

$$M_\infty = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}\ ,\ M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\ ,\ M_{-1} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\ ,\ M'_{-1} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}\ , \tag{2.8}$$

where M_{-1} is to be used if the monodromy around $u = -1$ is computed with a basepoint in the lower half u-plane and M'_{-1} if the basepoint is in the upper half u-plane. This distinction comes about since the neighbourhoods of $u = -1$ in the upper and lower half planes are separated by the branch cuts. This singularity structure corresponds to a massless magnetic monopole $(0, 1)$ at $u = 1$ and to a massless dyon, alternatively described as $(2, 1)$ or $(-2, 1)$, at $u = -1$.

The curve of marginal stability $\mathcal{C}_0$ was studied in [4,14], see Fig. 2. Along this curve, a_D/a takes all the values in the interval $[-2, 2]$, with a_D/a increasing monotonically from -2 to $+2$ as one goes along the curve clockwise from $u = -1 + i\epsilon$ to $u = -1 - i\epsilon$, with $a_D/a = 0$ at $u = 0$.

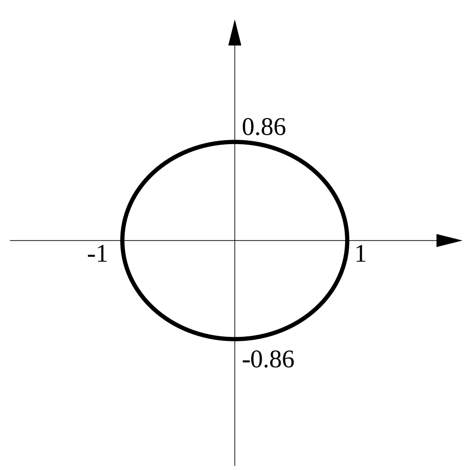

Fig. 2: The curve $\mathcal{C}$ in the complex u-plane where $\frac{a_D}{a} \in \mathbf{R}$.

The mass formula (2.1) implies the following important property: a BPS state with $\frac{n_e}{n_m} \equiv r \in [-2, 2]$ will become massles at the point $u^* \in \mathcal{C}_0$ where $\frac{a_D}{a}(u^*) = r$. On the other hand, the only singularities at finite u on the moduli space are at $u = \pm 1$, and hence the only BPS states ever getting massless are the magnetic monopole and the dyon. This constraint was extensively used in [1] in the derivation of the $N_f = 0$ spectra. In the $N_f > 0$ case the same constraint will also prove to be very useful.

Let us finally mention that for $N_f = 0$, with the present normalisation, the weak-coupling (semi-classical) spectrum consists of the dyons $(2n, 1)$, $n \in \mathbf{Z}$ as well as the W-bosons $(2, 0)$. The strong-coupling spectrum contains the monopole $(0, 1)$ and the dyon $(\pm 2, 1)$ only [1].

2.2. Global symmetries

Let us recall the field content of the theories we consider. In addition to the $N = 2$ vector multiplet containing the gluons, the two gluinos (λ, ψ) and the Higgs scalar ϕ, we have N_f hypermultiplets $(q^f, \chi^f, \tilde{q}_f, \tilde{\chi}_f)$ where $(q^f,\ \tilde{q}_f)$ are complex scalars and $(\chi^f,\ \tilde{\chi}_f)$ are Weyl spinors. f is the flavour index running from 1 to N_f and (q, χ) and $(\tilde{q}, \tilde{\chi})$ transform in complex conjugate representations of the gauge group. For $SU(2)$ these are actually equivalent representations.

Because $N = 2$ supersymmetry is not broken in this theory, we have an $SU(2)_R$ quantum symmetry for which the gluinos and the scalars $(q, \tilde{q}^\dagger)$ are doublets. This global exact quantum symmetry will play no particular rôle in the following. At the classical level, since the bare masses of the quarks are zero, we also have an $U(1)_R$ symmetry under which the gluinos have charge 1, the Weyl spinors in the hypermultiplets charge -1, and the Higgs scalar charge 2. This symmetry acts on the Higgs scalar and thus potentially on the moduli space. It will be of utmost importance in our analysis. Finally, when $N_f > 0$, which is the case on which we focus by now, we have the flavour symmetry which is exceptionally $O(2N_f)$ here, thanks to the pseudo-reality of $SU(2)$ representations. The $U(1)_R$ and $O(2N_f)$ symmetries are anomalous. Only a subgroup $\mathbf{Z}_{2(4-N_f)}$ of $U(1)_R$ survives quantum mechanically. Moreover, the parity ρ in $O(2N_f)$ is also anomalous, and the quantum flavour symmetry is really $SO(2N_f)$, or more precisely $Spin(2N_f)$, as the theory contains spinor multiplets of $SO(2N_f)$ (typically, semi-classical reasoning shows that this is the case for the particles with unit magnetic charge [6,10,15]). This parity ρ, which exchanges (q, χ) and $(\tilde{q}, \tilde{\chi})$ for one flavor of quark and leaves the

other fields unchanged, when composed with the $U(1)_R$ symmetry, gives rise to an improved exact $\mathbf{Z}_{4(4-N_f)}$ quantum symmetry, whose square is nothing but the $\mathbf{Z}_{2(4-N_f)}$ coming from $U(1)_R$ alone.

$\mathbf{Z}_4$ symmetry at fixed u

At a given point u of the Coulomb branch, this $\mathbf{Z}_{4(4-N_f)}$ symmetry is spontaneously broken by the vacuum expectation value of the Higgs field down to $\mathbf{Z}_4$. This $\mathbf{Z}_4$ changes the sign of the Higgs field and thus of the electric and magnetic charges. This explicitly shows that when the state (n_e, n_m) is present, then the anti-particle $(-n_e, -n_m)$ is also present. Thus we will often denote these simply by $\pm(n_e, n_m)$. However, note that, unlike for $N_f = 2$, for $N_f = 1$ and $N_f = 3$ this apparently innocent charge conjugation operation does include a ρ transformation. For $N_f = 1$, this simply changes the sign of the $U(1)$ flavour charge S. For $N_f = 3$, ρ acts as the outer automorphism of the corresponding Lie algebra (D_3 in the Cartan classification) and will change the $Spin(6)$ representation under which the (n_e, n_m) state transforms into its complex conjugate. For the spinor representations this amounts to changing the chirality. For $N_f = 2$, particles and anti-particles occur in the same representations of $Spin(4)$.

$\mathbf{Z}_{4-N_f}$ symmetry on the Coulomb branch

The spontaneous breakdown of the $\mathbf{Z}_{4(4-N_f)}$ to the $\mathbf{Z}_4$ symmetry is at the origin of the $\mathbf{Z}_{4-N_f}$ symmetry on the Coulomb branch of the moduli space. This is $\mathbf{Z}_3$ for $N_f = 1$ or $\mathbf{Z}_2$ for $N_f = 2$, where the broken generators will relate *different* points on the Coulomb branch where physically equivalent theories must lie. For $N_f = 3$, no such symmetry exists. The generator of this $\mathbf{Z}_3$ (for $N_f = 1$) or $\mathbf{Z}_2$ (for $N_f = 2$) symmetry always contains a ρ transformation and thus in particular will act on the flavour quantum numbers of the states. For $N_f = 1$ it changes the sign of the $U(1)$ flavour charge. For $N_f = 2$, the flavor group is $Spin(4) = SU(2) \times SU(2)$ and ρ has the effect of interchanging the two $SU(2)$ factor (this amounts to changing the chirality for the spinor representations). Note also that in all cases the generator of the symmetry acting on the Coulomb branch shifts the θ angle by π and thus should shift the electric charge n_e of a state (n_e, n_m) by n_m. This will be explicitly checked in the following.

Relations between the quantum numbers

We list below some constraints on the representation of the flavour group $Spin(2N_f)$ a BPS state (n_e, n_m) can carry. These constraints come essentially from the semi-classical

approach which allows one to obtain the general form of the wave function for a (n_e, n_m) state, valid in the regime where the field theory is weakly coupled. Semi-classical reasoning is likely to give the correct answer for the quantum numbers of the states (though not for others quantitative features like the physical mass). We will explain in Section 3 how this is corrected by non-perturbative effects. For $N_f = 1$, the flavour group is $SO(2) = U(1)$, and the abelian charge S under this $U(1)$ must be such that $2S = n_m - 2n_e \bmod 4$. Note that this charge S is not the abelian charge appearing in the central charge Z of the susy algebra when the bare masses of the hypermultiplets are non vanishing [6]. For $N_f = 2$, the flavour group is $Spin(4) = SU(2) \times SU(2)$ and the irreducible representations can be labeled like $(2s_1 + 1, 2s_2 + 1)$ where s_1 and s_2 are the spins of the corresponding $SU(2)$ representations. Then s_1 must be half integer if $n_e + n_m$ is odd, and must be integer if $n_e + n_m$ is even. The same is true for s_2 with n_e replacing $n_e + n_m$ (note that we could interchange the rôles of s_1 and s_2; the convention chosen here and below always correspond to the conventions of [6]). For $N_f = 3$ the flavour group is $Spin(6) = SU(4)$. If $2n_e + n_m$ is odd, we have a faithful representation of $SU(4)$. The cases $2n_e+n_m = 1 \bmod 4$ and $2n_e+n_m = 3 \bmod 4$ correspond to complex conjugate representations. We will choose that the monopole $(0, 1)$ is in the defining representation $\mathbf{4}$ of $SU(4)$. If $2n_e + n_m = 2 \bmod 4$ we have a faithful representation of $SO(6)$, and finally if $2n_e + n_m$ is a multiple of four we have a faithful representation of $SU(4)/\mathbf{Z}_4$, where here $\mathbf{Z}_4$ is the center of $SU(4)$, or the trivial representation.

We can be more precise for the states having unit magnetic charge. These are always in spinorial representations. A state $(n_e, +1)$ has $S = 1/2$ or $S = -1/2$ (for $N_f = 1$), it is in $(\mathbf{2}, \mathbf{1})$ or in $(\mathbf{1}, \mathbf{2})$ (for $N_f = 2$), and it is in $\mathbf{4}$ or in $\overline{\mathbf{4}}$ (for $N_f = 3$), where in each case the first entry corresponds to n_e even and the second entry to n_e odd. Concerning the S charge of the quarks for $N_f = 1$, note that half of the states $(+1, 0)$ have $S = +1$, and the other half have $S = -1$.

All these constraints are perfectly consistent with the former analysis of the action of the global symmetries. Here we check this only for $N_f = 3$, and leave the other cases to the reader. Note that $2n_e + n_m = 1 \bmod 4$ is equivalent to $-2n_e - n_m = 3 \bmod 4$, and thus that charge conjugation amounts to complex conjugating the representations in this case, as it should. The cases $2n_e + n_m = 0 \bmod 4$ and $2n_e + n_m = 2 \bmod 4$ are left invariant by charge conjugation; this again is consistent since a faithful representation of $SO(6)$ or $SU(4)/\mathbf{Z}_4$ gives another

faithful representation of the corresponding groups by complex conjugation. Let us now check the action of the generator of the $\mathbf{Z}_3$ symmetry. It indeed acts as charge conjugation as it contains a ρ transformation, since $2(n_e + n_m) + n_m = -2n_e - n_m$ mod 4.

2.3. $\mathbf{Z}_{4-N_f}$ symmetry and BPS states

In this subsection, we will rephrase and sharpen some of the arguments already mentioned in Section 4.1 of [1]. The supersymmetry charges, which we will denote by $Q_{\alpha,I}$ in chiral notation ($1 \leq I \leq 2$ for $N = 2$ supersymmetry), have charge one under the $\mathbf{Z}_{4(4-N_f)}$ symmetry. The relevant piece of the supersymmetry algebra for our purposes is

$$\{Q_{\alpha,I}, Q_{\beta,J}\} = 2\,\epsilon_{\alpha\beta}\,\epsilon_{IJ}\,Z\ . \tag{2.9}$$

This shows that the central charge $Z = an_e - a_D n_m$ has charge two. Z can be charged under a global discrete symmetry group, since such a group is not generated by infinitesimal generators belonging to the algebra. The transformation law of Z under the action of the generator of the $\mathbf{Z}_{4-N_f}$ symmetry acting on the moduli space is then

$$Z \longrightarrow Z' = e^{\pm \frac{i\pi}{4-N_f}}\,Z\ . \tag{2.10}$$

Thus we have $|Z'| = |Z|$. Moreover, since the $\mathbf{Z}_{4-N_f}$ symmetry of course maps states of the same physical mass into each other ($m' = m$), we see that a BPS state will be mapped into another BPS state: $m = \sqrt{2}\,|Z|$ implies $m' = \sqrt{2}\,|Z'|$. This fact could also be understood in terms of conservation of the number of degrees of freedom, as a BPS state lies in a short representation of the supersymmetry algebra unlike the other states with $m > \sqrt{2}\,|Z|$. Moreover, if u and u' denote two points related by $\mathbf{Z}_{4-N_f}$, and if there exists a matrix $G \in SL(2,\mathbf{Z})$ and a phase $e^{i\omega}$ determined for instance by (2.10) such that

$$\begin{pmatrix} a_D \\ a \end{pmatrix}(u') = e^{i\omega}\,G \begin{pmatrix} a_D \\ a \end{pmatrix}(u)\ ,\ \begin{pmatrix} n'_e \\ n'_m \end{pmatrix} = G \begin{pmatrix} n_e \\ n_m \end{pmatrix}\ , \tag{2.11}$$

the relation $|Z'| = |Z|$ will be obviously satisfied. We will show in the following that relation (2.11) is indeed realized.

To summarize, the existence of a BPS state $p = (n_e, n_m)$ at u implies the existence of another BPS state $p' = (n'_e, n'_m)$ at u', u and u' being related by a global symmetry acting on the moduli space, with the following properties: they have the same mass, their electric and magnetic quantum numbers are related as in (2.11), and the representations of the flavour group in which they lie, though they may be different as explained in Section 2.2, have the same dimension.

2.4. A note on some mild assumptions made in this work and some of their consequences

On the Coulomb branch, the coordinate u labels physically inequivalent theories. The operators $\mathcal{O}$ corresponding to the observables of the theory at u may thus be labeled by u, and act in a Hilbert space $\mathcal{H}_u$. It is natural to suppose that, at least locally, $\mathcal{H}_u$ does not depends on u and that the eigenvalues and eigenfunctions of the operators $\mathcal{O}_u$ vary continuously with u. Thus, the mass $m(u)$ of a BPS state, as given in (2.1), must be a continuous function of u as one moves in the Coulomb branch. This should also be the case for the electric and magnetic charges, but as these are integers, they must be constant. For the same reason, the Witten index or its generalizations, used to show that supersymmetry is not broken in the models we consider, must be constant.

This picture cannot be maintained globally. The reason for this is that there exist curves of marginal stability, separating strong and weak-coupling regions, allowing the "decays" of usually stable BPS states. In [1], it was shown that such decays *do* happen in the pure gauge theory, and one of the most important results of the present paper will be to explicitly demonstrate that this is also the case for the $1 \leq N_f \leq 3$ theories. Thus, at least in the BPS sector of the Hilbert space, there must be some kind of discontinuity on this curve. The transition between the two sides of the curve may be understood as follows. The theories which lie just on the curve should have physically equivalent sectors in their Hilbert space. The predicted "decay" patterns on the curve are nothing but the rules for this identification. If this is correct, the quantum numbers of the states related by such a decay should be compatible, and we will see that this is always the case. Once the identification is made, one can forget about the different sectors and only work in a reduced Hilbert space which corresponds to one given sector. Now, when we leave the curve, moving into the strong-coupling region, we are

left with only the reduced Hilbert space! Typically, the BPS sector of this reduced Hilbert space will contain only the states which are responsible for the singularities on the Coulomb branch. Note that it is not clear whether the Hilbert space contains other sectors than the BPS sector, and if this hypothetic new sectors can undergo discontinuities.

Another interesting feature of the theories under study is the following. If a spontaneously broken global symmetry relates two points u and u' which can be linked by a path which does not cross any discontinuity curve, then the *broken* symmetry predicts the existence of certain states in a theory at *fixed* u. We will show that, in the strong-coupling region, this implies that the BPS states come in multiplets of the broken $\mathbf{Z}_{4(4-N_f)}$ symmetry! In the $N_f = 0$ case we had the $\mathbf{Z}_2$ pairs of [1], and here we will have $\mathbf{Z}_3$ pairs for $N_f = 1$ and $\mathbf{Z}_2$ pairs for $N_f = 2$.

3. One flavour of quarks

3.1. Global analytic structure, and $\mathbf{Z}_3$ symmetry

We begin by discussing the case of a single massless hypermultiplet. We present here all the arguments in quite some detail. When dealing with $N_f = 2$ and $N_f = 3$ in the following sections, we will mainly focus on the new features and try to avoid repetitions.

As was argued in [6], for $N_f = 1$ one has three singularities at finite u, related by the $\mathbf{Z}_3$ symmetry, and due to dyons becoming massless. We will choose the scale and orientation on the u-plane so that they are located at the points $u_1 = e^{i\pi/3}$, $u_2 = -1$ and $u_3 = e^{-i\pi/3}$. Suppose that the singularity at u_3 is produced by a state (n_e, n_m). The associated monodromy matrix is given by (2.4) with $d = 1$ (here the flavour group is abelian and its irreducible representations are all of dimension 1). According to the discussion in Section 2.2, because of the $\mathbf{Z}_3$ symmetry the states becoming massless at u_1 and u_2 will be respectively $(n_e - n_m, n_m)$ and $(n_e - 2n_m, n_m)$. A general consistency condition for the monodromy group is actually

$$M_{(n_e-2n_m,n_m),1}M_{(n_e-n_m,n_m),1}M_{(n_e,n_m),1} = M_\infty = M_{(n_e-n_m,n_m),1}M_{(n_e,n_m),1}M_{(n_e+n_m,n_m),1}, \tag{3.1}$$

where M_∞ is given by (2.3). This implies $n_m = \pm 1$. The electric charge is left arbitrary, reflecting the possibility of "democracy" transformations, see [1], and we choose $n_e = 0$. This shows that the "minimal" choices of [6] are actually the only possible one.

Let us now determine the analytic structure of the $SL(2,\mathbf{Z})$ bundle E_1. The corresponding elliptic curve is [6]

$$y^2 = x^3 - ux^2 - \frac{\Lambda_1^6}{64} \ , \tag{3.2}$$

and $\mathrm{d}a_D^{(1)}/\mathrm{d}u$ and $\mathrm{d}a^{(1)}/\mathrm{d}u$ are given by its period integrals. Then $a_D^{(1)}$ and $a^{(1)}$ are given by the integrals over appropriate contours γ_i of a one-form $\tilde{\lambda}$, where $\tilde{\lambda}$ is obtained by integrating $\frac{\sqrt{2}}{8\pi}\frac{\mathrm{d}x}{y}$ with respect to u (up to an exact one-form on the elliptic curve). This yields the integrals

$$\frac{\sqrt{2}}{8\pi}\oint_{\gamma_i} \mathrm{d}x \frac{2u - 3x}{\sqrt{x^3 - ux^2 - \Lambda_1^6/64}} \tag{3.3}$$

for $a_D^{(1)}$ and $a^{(1)}$. The most convenient choice for the mass scale, leading to singularities on $|u| = 1$, is such that

$$\Lambda_1^6 = \frac{256}{27}. \tag{3.4}$$

It is straightforward to show from (3.3), and it was established in [11] that $a_D^{(1)}$ and $a^{(1)}$ satisfy the differential equation*

$$\left[(u^3+1)\frac{\mathrm{d}^2}{\mathrm{d}u^2} - \frac{u}{4}\right]\begin{pmatrix} a_D^{(1)} \\ a^{(1)} \end{pmatrix}(u) = 0 \ . \tag{3.5}$$

This is not a hypergeometric equation, but it can be transformed into one if we change variables [11] to

$$v = -u^3 \ . \tag{3.6}$$

Then

$$\left[v(1-v)\frac{\mathrm{d}^2}{\mathrm{d}v^2} + \frac{2}{3}(1-v)\frac{\mathrm{d}}{\mathrm{d}v} - \frac{1}{36}\right]\begin{pmatrix} a_D^{(1)} \\ a^{(1)} \end{pmatrix} = 0 \ . \tag{3.7}$$

This is a hypergeometric differential equation with $a = b = -\frac{1}{6}$ and $c = \frac{2}{3}$.

$\star$ It comes from the Picard-Fuchs equation satisfied by the period integrals which are the derivatives of $a_D^{(1)}$ and $a^{(1)}$ with respect to u.

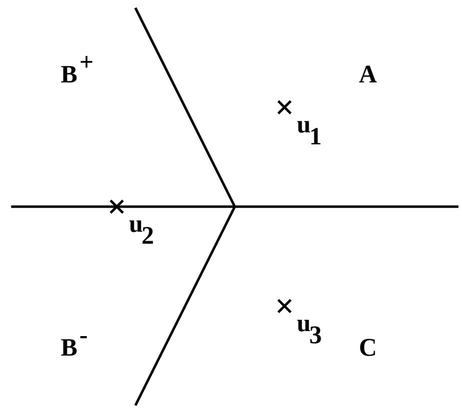

Fig. 3: Definition of the regions A, B^+, B^-, C and positions of the singularities u_1, u_2, u_3 for $N_f = 1$

Note that the mapping (3.6) is 3 to 1 (except at $u = 0$) and each of the three regions A, B ($\equiv B^+ \cup B^-$) and C of the u-plane shown in Fig. 3 are mapped onto the full v-plane. The singular points of the differential equation (3.7) are $v = 0, 1$ and ∞, corresponding in the u-plane to

$$u_0 = 0 \ , \ u_1 = e^{i\pi/3} \ , \ u_2 = -1 \ , \ u_3 = e^{-i\pi/3} \ , \ u_\infty = \infty \ . \tag{3.8}$$

The borders between the regions A, B and C are mapped onto the negative real v-axis. Any solution of eq. (3.7) on the v-plane will yield a solution of (3.5) in any one of the three regions A, B or C. However, to obtain a solution on the whole u-plane we must choose some solution in a given region, say in C, and then analytically continue it into the neighbouring regions. Note also that although $v = 0$ is a singular point of (3.7), $u = 0$ of course is not a singular point of (3.5).

A convenient choice of two linearly independent solutions to the differential equation (3.7) are Kummer's solutions $\tilde{U}_3(v)$ and $U_6(v)$ [13]:

$$\begin{aligned} U_6(v) &= (1-v) F\left(\frac{5}{6}, \frac{5}{6}, 2; 1-v\right) \\ \tilde{U}_3(v) &= v^{1/6} F\left(-\frac{1}{6}, \frac{1}{6}, 1; \frac{1}{v}\right) \ . \end{aligned} \tag{3.9}$$

The first solution $U_6(v)$ obviously vanishes as $v \to 1$, and hence, once appropriately normalized,

is a good candidate for $a_D^{(1)}$. The second solution behaves for $v \to \infty$ as

$$v^{1/6} = (-u^3)^{1/6} = \omega(u)\sqrt{u} \ , \tag{3.10}$$

where $\omega(u)$ is a region-dependent phase factor: $\omega(u) = e^{-i\pi/2}$ for $u \in B^+$, $\omega(u) = e^{-i\pi/6}$ for $u \in A$, $\omega(u) = e^{i\pi/6}$ for $u \in C$ and $\omega(u) = e^{i\pi/2}$ for $u \in B^-$. Once correctly normalized, $\tilde{U}_3$ is thus a good candidate for $a^{(1)}$. The asymptotics (3.10) are discontinuous, but as explained above, one should take the solution in one region and then analytically continue it to the other regions. The correctly normalized function $a^{(1)}$ then is

$$a^{(1)}(u) = \frac{1}{2}\sqrt{2u}\, F\left(-\frac{1}{6},\frac{1}{6},1;-\frac{1}{u^3}\right) \ . \tag{3.11}$$

This function has one branch cut along the negative real u-axis from the square-root, as well as three other cuts extending from the origin of the u-plane to u_1, u_2 and u_3 due to the hypergeometric function, see Fig. 4. The expression (3.11) for $a^{(1)}(u)$ obviously has the correct asymptotics $\sim \frac{1}{2}\sqrt{2u}$ everywhere as $u \to \infty$.

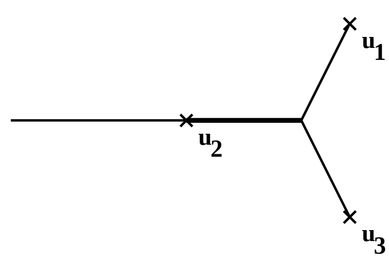

Fig. 4: The cuts of $a^{(1)}(u)$

Next, we determine $a_D^{(1)}(u)$. It should be proportional to $U_6(v)$ as given in (3.9) in the region A or B or C in which the magnetic monopole becomes massless. Choosing this region is a matter of convention, just as for $N_f = 0$ in [5] it was chosen to have the massless monopole at $u = 1$ (rather than at $u = -1$). We choose this region to be C with the monopole becoming massless at $u_3 = e^{-i\pi/3}$. The normalisation of $a_D^{(1)}$ then is determined from the

desired asymptotics (2.2). If we define for all u the function

$$f_D(u) = \frac{\sqrt{2}}{12}\left(u^3+1\right) F\left(\frac{5}{6},\frac{5}{6},2;1+u^3\right) \tag{3.12}$$

then for $u \in C$ we have

$$a_D^{(1)}(u) = e^{-2i\pi/3} f_D(u)\ , \quad u \in C\ . \tag{3.13}$$

Indeed, as $u \to \infty$ in region C, the asymptotics of this expression is

$$\frac{3i}{4\pi}\sqrt{2u}\left[\frac{1}{3}\log(-u^3)+\frac{4}{3}\log 2 - 2 + \log 3\right] = \frac{3i}{4\pi}\sqrt{2u}\left[\log u + \frac{4}{3}\log 2 - 2 + \log 3 + \frac{i\pi}{3}\right]\ , \tag{3.14}$$

in accordance with the asymptotics derived directly from the integral (3.3) [11]. One finds for all analytic continuations

$$\begin{aligned}
u \in B^- &: \quad a_D^{(1)}(u) = -f_D(u) + a^{(1)}(u)\ ,\\
u \in C &: \quad a_D^{(1)}(u) = e^{-2i\pi/3} f_D(u)\ ,\\
u \in A &: \quad a_D^{(1)}(u) = e^{-i\pi/3} f_D(u) - a^{(1)}(u)\ ,\\
u \in B^+ &: \quad a_D^{(1)}(u) = +f_D(u) - 2a^{(1)}(u)\ .
\end{aligned} \tag{3.15}$$

We recall that $a^{(1)}(u)$ was given in (3.11) for all u. Since the different expressions for $a_D^{(1)}(u)$ are obtained by analytic continuation through the cuts of $F\left(\frac{5}{6},\frac{5}{6},2;1+\frac{4}{27}u^3\right)$, which are the borders between regions B^+ and A, A and C, C and B^-, the function $a_D^{(1)}(u)$ so defined has no cuts at these borders. The only such cut at the border between two regions is between B^+ and B^-. Moreover, in region A, due to the presence of $a^{(1)}(u)$ in the definition of $a_D^{(1)}(u)$, there is also a cut from $u=0$ to $u=u_1$. The cuts of $a_D^{(1)}(u)$ are shown in Fig. 5.

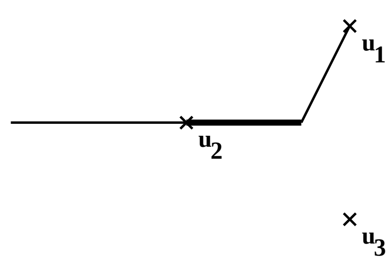

Fig. 5: The cuts of $a_D^{(1)}(u)$

The solutions (3.15) reflect the $\mathbf{Z}_3$ symmetry on the moduli space. One has

$$\begin{pmatrix} a_D^{(1)} \\ a^{(1)} \end{pmatrix} \left(e^{\pm 2i\pi/3}u\right) = e^{\pm i\pi/3} G_{W\pm} \begin{pmatrix} a_D^{(1)} \\ a^{(1)} \end{pmatrix} (u) \quad , \quad G_{W\pm} = \begin{pmatrix} 1 & \mp 1 \\ 0 & 1 \end{pmatrix} \quad , \tag{3.16}$$

provided that u is such that the path $t \mapsto e^{\pm 2i\pi t/3}u,\ t \in [0,1]$ does not cross the cut on the negative real u-axis, see Fig. 6.

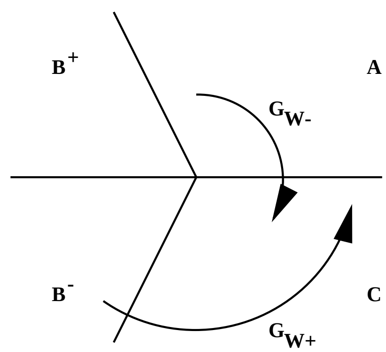

Fig. 6: The matrices $e^{\pm i\pi/3}G_{W\pm}$ provide the analytic continuations associated with the $\mathbf{Z}_3$ symmetry.

We now proceed to give the various asymptotics and monodromy matrices for the singular points. First, of course, $u = 0$ is not a singular point, as we already noticed from the differential equation (3.5), although this is not completely manifest on our solution. However, for $u \to 0$, $a_D^{(1)}(u)$ and $a^{(1)}(u)$ have a power series development of the form $c_0 + c_1 u + \ldots$, *without* any logarithms. The constants $c_0, c_1, \ldots$ have different phases on different sides of the cuts going through $u = 0$, as shown in Fig. 4 and 5. Nevertheless, if one takes the asymptotic form at some point u close to 0 and analytically continues it along a path surrounding $u = 0$ one obviously gets back the same series, and the monodromy is trivial, as it should. For the other asymptotics we find:

$$\left.\begin{aligned} a^{(1)}(u) &\simeq \frac{1}{2}\sqrt{2u} \\ a_D^{(1)}(u) &\simeq \frac{3i}{4\pi}\sqrt{2u}\left[\log u + \frac{4}{3}\log 2 - 2 + \log 3 + \frac{i\pi}{3}\right] \end{aligned}\right\} \quad \text{as } u \to \infty$$

$$\left.\begin{aligned}
a^{(1)}(u) &\simeq \frac{\sqrt{2}}{8\pi} e^{5i\pi/6} \Big[12e^{-2i\pi/3} + (u-u_1)\log(u-u_1) \\
&\qquad + (u-u_1)\left(5 - 4\log 2 - 2\log 3 - \frac{i\pi}{3}\right)\Big] \\
a_D^{(1)}(u) &\simeq -a^{(1)}(u) + \frac{\sqrt{2}}{4} e^{i\pi/3}(u-u_1)
\end{aligned}\right\} \quad \text{as } u \to u_1$$

$$\left.\begin{aligned}
a^{(1)}(u) &\simeq \frac{\sqrt{2}}{8\pi} i\epsilon \Big[12 + (u-u_2)\log(u-u_2) \\
&\qquad + (u-u_2)\left(5 - 4\log 2 - 2\log 3 + i\pi\epsilon\right)\Big] \\
a_D^{(1)}(u) &\simeq -\frac{3\epsilon+1}{2} a^{(1)}(u) + \frac{\sqrt{2}}{4}\,\epsilon\,(u-u_2)
\end{aligned}\right\} \quad \text{as } u \to u_2$$

$$\left.\begin{aligned}
a^{(1)}(u) &\simeq \frac{\sqrt{2}}{8\pi} e^{-5i\pi/6} \Big[12e^{2i\pi/3} + (u-u_3)\log(u-u_3) \\
&\qquad + (u-u_3)\left(5 - 4\log 2 - 2\log 3 + \frac{i\pi}{3}\right)\Big] \\
a_D^{(1)}(u) &\simeq \frac{\sqrt{2}}{4} e^{2i\pi/3}(u-u_3)
\end{aligned}\right\} \quad \text{as } u \to u_3 \tag{3.17}$$

where ϵ is the sign of $\Im m\, u$. The monodromy matrices follow as

$$M_\infty = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix}, \ M_{u_1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \ M_{u_3} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$M_{u_2} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \ M'_{u_2} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}, \tag{3.18}$$

where M_{u_2} is to be used for a basepoint in the lower half u-plane ($\epsilon = -1$) and M'_{u_2} for a basepoint in the upper half u-plane ($\epsilon = +1$). We have (cf. (3.1)) $M_{u_1}M_{u_3}M_{u_2} = M_\infty = M'_{u_2}M_{u_1}M_{u_3}$. As a check one verifies from (3.17) that the particles that become massless are: the magnetic monopole $(0,1)$ at $u = u_3$, the dyon $(-1,1)$ at $u = u_1$ and, at $u = u_2$, the dyon described as $(-2,1)$ if $\epsilon = +1$ or as $(1,1)$ if $\epsilon = -1$.

3.2. The curve of marginal stability

The curve of marginal stability $\mathcal{C}_1$ is defined as the set of all u on the moduli space such that $w^{(1)}(u) \equiv a_D^{(1)}(u)/a^{(1)}(u)$ is real. We already know that this curve has to pass through u_1, u_2 and u_3, since massless particles must occur on this curve, and we have $w^{(1)}(u_3) = 0$, $w^{(1)}(u_1) = -1$, etc. As for $N_f = 0$, one can try to determine the curve $\mathcal{C}_1$ analytically as follows: the Picard-Fuchs differential equation (3.5) implies that $w^{(1)}$ satisfies

$$\{w^{(1)}, u\} = -2\frac{u}{u^3+1} \tag{3.19}$$

where $\{f, u\} = f'''/f' - \frac{3}{2}\left(f''/f'\right)^2$ denotes the Schwarzian derivative of f with respect to u. The inverse function $u(w^{(1)})$ then satisfies the differential equation

$$\frac{u'''}{u'} - \frac{3}{2}\left(\frac{u''}{u'}\right)^2 = 2(u')^2\frac{u}{u^3+1} \tag{3.20}$$

and it is enough to find the solution for a real argument $w^{(1)}$ that satisfies the appropriate initial conditions $u(0) = u_3$, $u(-1) = u_1$, etc.

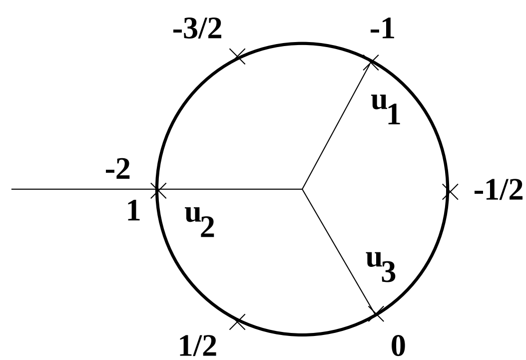

Fig. 7: The curve of marginal stability for $N_f = 1$ passes through the three cubic roots u_1, u_2, u_3 of -1. It is almost a circle. The numbers $-2, -3/2, \ldots, 1$ indicate the values taken by $w^{(1)} = a_D^{(1)}/a^{(1)}$ along this curve.

But again, the precise determination of the curve $\mathcal{C}_1$ is completely irrelevant for our purpose. Instead, just as we did in [1], we have determined $\mathcal{C}_1$ numerically, directly from the condition $\Im m\, w^{(1)}(u) = 0$. In any case, the $\mathbf{Z}_3$ symmetry (3.18) implies that this curve is

symmetric under rotations by $\frac{2\pi}{3}$ around the origin. It turns out to be almost a circle,* see Fig. 7.

If we call $\mathcal{C}^-$ the portion of $\mathcal{C}_1$ between u_2 and u_3 (see Fig. 8), $\mathcal{C}^0$ the portion between u_3 and u_1 and $\mathcal{C}^+$ the portion between u_1 and u_2, one can show that

$$u \in \mathcal{C}^+ \;\Leftrightarrow\; w^{(1)} \in [-2,-1] \quad , \quad u \in \mathcal{C}^0 \;\Leftrightarrow\; w^{(1)} \in [-1,0] \quad , \quad u \in \mathcal{C}^- \;\Leftrightarrow\; w^{(1)} \in [0,1] \tag{3.21}$$

with $w^{(1)}(u)$ increasing monotonically as u goes around the curve in the clockwise sense starting at u_2.

3.3. The weak-coupling spectrum

Let us first determine the spectrum of BPS states in the weak-coupling region $\mathcal{R}_W$, i.e. outside the curve $\mathcal{C}$.

On the one hand we know that there are the elementary excitations of the perturbative spectrum, namely the W bosons $(2,0)$ and the quarks $(1,0)$. Note that it seems that the W are degenerate with two quarks and thus may disintegrate as $(2,0) \rightarrow 2\times(1,0)$. However this reaction is impossible here since the W have flavor charge $S=0$ while the quarks $(1,0)$ have $S=1$. Thus the W must be stable in this theory. In the cases $N_f=2$ and $N_f=3$, such a simple argument does not work, and we will then admit the existence of the W in the weak-coupling regions, interpreting these states as bound states at threshold, as was done in [6]. Clearly, $\pm(1,0)$ and $\pm(2,0)$ are the only stable states of zero magnetic charge. Now, let's turn to the magnetically charged states. Since the three states $(0,1)$, $(-1,1)$ and $(-2,1)$ become massless at u_3, u_1 and u_2, i.e. just on the curve $\mathcal{C}_1$, they must exist in both $\mathcal{R}_W$ and $\mathcal{R}_S$. The monodromy around infinity is a symmetry of the semi-classical weak-coupling spectrum $\mathcal{S}_W$, so we conclude that all states $\pm(n_e-3k,1)$, $n_e=0,1,2$, $k\in\mathbf{Z}$ must be in $\mathcal{S}_W$. But these are all the dyons $\pm(n,1)$, $n\in\mathbf{Z}$, with unit magnetic charge. It is also easy to see that one cannot have dyons with $|n_m|\geq 2$ in $\mathcal{S}_W$. If there were such a dyon $(n_e,n_m)\in\mathcal{S}_W$ then, again, all dyons $(n_e-3k\,n_m,n_m)$ would be in $\mathcal{S}_W$, too. Such a dyon would become massless on $\mathcal{C}_1$ and lead to a singularity if $\dfrac{n_e-3kn_m}{n_m}=\dfrac{n_e}{n_m}-3k\in[-2,1]$. There is always such a $k\in\mathbf{Z}$. But

* The distance from the origin is slightly larger at $u=u_1,u_2,u_3$ and slightly smaller at the borders between the regions A,B,C.

we know that the only massless dyons have $n_m = 1$. Hence there are no dyons with $|n_m| \geq 2$ in $\mathcal{S}_W$, and

$$\mathcal{S}_W = \{\pm(2,0)\ ,\ \pm(1,0)\ ,\ \pm(n,1)\ ,\ n \in \mathbf{Z}\}\ . \tag{3.22}$$

Finally, recall from Sect. 2.2 that the states $\pm(2,0)$ have $S = 0$, half of the quark states $\pm(1,0)$ have $S = \pm 1$ and the other half have $S = \mp 1$. Concerning the dyons, $\pm(2n,1)$ have $S = \pm 1/2$, and $\pm(2n+1,1)$ have $S = \mp 1/2$.

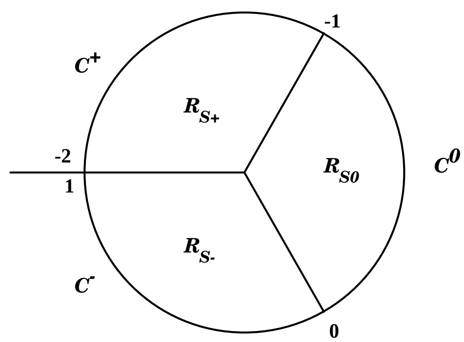

Fig. 8: The definition of the various portions of the curve and of the strong-coupling region

3.4. The strong-coupling spectrum

Mathematical description of BPS states

Next, we turn to the strong-coupling spectrum. As for $N_f = 0$ [1], due to the cuts, see Fig. 8, the *same* BPS state is described by different couples of integers in the different regions $\mathcal{R}_{S+}$, $\mathcal{R}_{S-}$, $\mathcal{R}_{S0}$. We call $\mathcal{S}_{S+}$, $\mathcal{S}_{S-}$, $\mathcal{S}_{S0}$ the corresponding spectra of integers (n_e, n_m), while $\mathcal{S}_S$ means the corresponding spectrum of BPS states (unique throughout $\mathcal{R}_S$). Let's first work out these different descriptions. Suppose we have a state described in $\mathcal{R}_{S0}$ by (n_e, n_m). Now transport it along a path inside the strong-coupling region $\mathcal{R}_S$ to a point $u' \in \mathcal{R}_{S-}$. Since one does not cross the curve $\mathcal{C}$, the state remains stable and cannot decay. The mass $m(u) = \sqrt{2}\left|n_e a^{(1)}(u) - n_m a_D^{(1)}(u)\right|$ will vary continuously, even as one crosses the cut, since physically nothing happens there. What happens as one crosses the cut is that one is

analytically continuing $a_D^{(1)}(u)$ and $a^{(1)}(u)$ to some $\tilde{a}_D^{(1)}(u'),\ \tilde{a}^{(1)}(u')$. The latter are expressed in terms of $a_D^{(1)}(u'),\ a^{(1)}(u')$ by using the monodromy matrix around u_3 as follows:

$$\begin{pmatrix} \tilde{a}_D^{(1)} \\ \tilde{a}^{(1)} \end{pmatrix}(u') = M_{u_3} \begin{pmatrix} a_D^{(1)} \\ a^{(1)} \end{pmatrix}(u') \ . \tag{3.23}$$

Hence

$$m(u') = \sqrt{2}\left| \eta\big((n_e, n_m), (\tilde{a}_D^{(1)}, \tilde{a}^{(1)})(u')\big)\right| = \sqrt{2}\left|\eta\big(M_{u_3}^{-1}(n_e, n_m), (a_D^{(1)}, a^{(1)})(u')\big)\right| \tag{3.24}$$

and we see that the same BPS state is described in $\mathcal{R}_{S-}$ by

$$\begin{pmatrix} n'_e \\ n'_m \end{pmatrix} = M_{u_3}^{-1} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} n_e \\ n_m + n_e \end{pmatrix} \quad \text{in } \mathcal{R}_{S-} \ . \tag{3.25}$$

Similarly, in $\mathcal{R}_{S+}$ this state is described by

$$\begin{pmatrix} n''_e \\ n''_m \end{pmatrix} = M_{u_1} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} 2n_e + n_m \\ -n_e \end{pmatrix} \quad \text{in } \mathcal{R}_{S+} \ . \tag{3.26}$$

In other words,

$$\mathcal{S}_{S-} = M_{u_3}^{-1}\mathcal{S}_{S0} \ , \ \mathcal{S}_{S0} = M_{u_1}^{-1}\mathcal{S}_{S+} \ , \ \mathcal{S}_{S+} = M_{u_1}M_{u_3}\mathcal{S}_{S-} \tag{3.27}$$

that is to say

$$p \equiv (n_e, n_m) \in \mathcal{S}_{S0} \quad \Longleftrightarrow \quad p \equiv \pm(n_e, n_m{+}n_e) \in \mathcal{S}_{S-} \quad \Longleftrightarrow \quad p \equiv \pm(2n_e{+}n_m, -n_e) \in \mathcal{S}_{S+} \ , \tag{3.28}$$

where p is a unique locally constant section over E_1.

These results deserve some comments. First note that the sign of the couple of integers representing p in $\mathcal{S}_{S-}$ and $\mathcal{S}_{S+}$ is not always fixed[⋆] since eq. (3.24) does not fix the sign in (3.25) or (3.26). However, in some cases we can determine this sign. For example, the

⋆ The sign is relevant here since the particle and anti-particle carry opposite flavour charge.

monopole state $(0,1)$ becoming massless at u_3 can be followed continuously from the weak-coupling to the strong-coupling region if one crosses the curve of marginal stability exactly at u_3 where $(0,1)$ is the only charged massless particle and thus is stable. Thus $(0,1)$ must be described by the same couple of integers in $\mathcal{S}_{S0}$ and in $\mathcal{S}_{S-}$. The same reasoning applies also to the dyon $(-1,1)$ at u_1. Now, the monopole $p \equiv (0,1)$ need not to be represented again by the same electric and magnetic quantum numbers in $\mathcal{S}_{S+}$. Indeed, (3.28) gives $p \equiv \pm(1,0)$ in $\mathcal{S}_{S+}$. This shows that the distinction between electric and magnetic quantum numbers in the strong-coupling region is not clear. This is a highly non perturbative phenomenon, and is possible due to the non-abelian monodromies. A somewhat similar phenomenon was discussed in [6] when considering the deformation of the singularities of the theory with non zero bare mass as the latter goes to zero. Nevertheless, the "monopole" $(1,0)$ in $\mathcal{S}_{S+}$ should not be confused with the elementary quark. Actually, $(1,0)$ in $\mathcal{S}_{S+}$ has $|S| = 1/2$ while an elementary quark has $|S| = 1$. The fact that a state $(1,0)$ can have $|S| = 1/2$ does not contradict the semi-classical constraint $2S = n_m - 2n_e$ mod 4, since the strong-coupling region is not continuously related to the semi-classical one. The only case where the semi-classical constraints on the quantum numbers are also valid in the strong-coupling region is when one can go through the curve $\mathcal{C}_1$ without any discontinuity as explained above (i.e. in $\mathcal{S}_{S0}$ and $\mathcal{S}_{S-}$ for the monopole, but not in $\mathcal{S}_{S+}$). Actually, when the strong-coupling region needs only be separated in two different charts, as is the case for $N_f = 2$ or 3, this argument implies that the transition functions of the $SL(2,\mathbf{Z})$ bundle are compatible with the semi-classical formula for the representations, as we will see in Section 4 and 5. But for $N_f = 1$ we need three charts and at this stage we are let with an ambiguity in the signs of (3.28) for general (n_e, n_m). For instance, we do not know if $(0,+1)$ in $\mathcal{S}_{S0}$ and $\mathcal{S}_{S-}$ is represented by $(+1,0)$ of $(-1,0)$ in $\mathcal{S}_{S+}$, and thus if the flavour charge of $(+1,0)$ in $\mathcal{S}_{S+}$ is $+1/2$ of $-1/2$. We will see in Section 3.5 how to lift this ambiguity.

Determination of the strong-coupling spectrum

Now we can determine the strong-coupling spectrum using the $\mathbf{Z}_3$ symmetry. We will work in $\mathcal{S}_{S0}$ and then obtain $\mathcal{S}_{S\pm}$ simply from (3.27). So let $u \in \mathcal{R}_{S0}$. Recall that by the general arguments of section 2.3, there must exist a matrix $G_{W+} \in SL(2,\mathbf{Z})$, satisfying (2.11) for $u' = e^{2\pi i/3}u$. This matrix was explicitly determined in (3.16). In section 2.3, it was shown that the existence of a BPS state (n_e, n_m) at $u \in \mathcal{R}_{S0}$ then implies the existence of a state $G_{W+}(n_e, n_m) \in \mathcal{S}_{S+}$ at $u' \in \mathcal{R}_{S+}$. This same BPS state must also exist at any point

in $\mathcal{R}_{S0}$, but there it is described by $M_{u_1}^{-1} G_{W+}(n_e,n_m) \in \mathcal{S}_{S0}$, according to (3.27). We define $G_{S+} = M_{u_1}^{-1} G_{W+}$ and conclude that $(n_e,n_m) \in \mathcal{S}_{S0}$ implies $G_{S+}(n_e,n_m) \in \mathcal{S}_{S0}$. In exactly the same way one can use the $\mathbf{Z}_3$ symmetry with $u' = e^{-2\pi i/3} u$ and conclude that there must also be a state $M_{u_3} G_{W-}(n_e,n_m) \in \mathcal{S}_{S0}$. Hence

$$G_{S+} = M_{u_1}^{-1} G_{W+} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \ , \quad G_{S-} = M_{u_3} G_{W-} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = G_{S+}^{-1} \ ,$$

$$G_{S+} \mathcal{S}_{S0} = G_{S-} \mathcal{S}_{S0} = \mathcal{S}_{S0} \ , \tag{3.29}$$

i.e. $\mathcal{S}_{S0}$ must be invariant under G_{S+}^p for any integer p. Since

$$G_{S\pm}^3 = -\mathbf{1} \ , \tag{3.30}$$

all strong-coupling states come in $\mathbf{Z}_3$ triplets:

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \overset{G_{S+}}{\longrightarrow} \begin{pmatrix} -n_m \\ n_e + n_m \end{pmatrix} \overset{G_{S+}}{\longrightarrow} \begin{pmatrix} -n_e - n_m \\ n_e \end{pmatrix} \overset{G_{S+}}{\longrightarrow} -\begin{pmatrix} n_e \\ n_m \end{pmatrix} . \tag{3.31}$$

Remark that these $\mathbf{Z}_3$ triplets contain actually six states when one differentiates particles and anti-particles. They are multiplet of the broken $\mathbf{Z}_{12}$ symmetry, as announced in Section 2.4. For example, the $\mathbf{Z}_3$ triplet containing the magnetic monopole $(0,1) \in \mathcal{S}_{S0}$ also contains $(1,0) \in \mathcal{S}_{S0}$, corresponding to $\pm(2,-1) \in \mathcal{S}_{S+}$ or to $\pm(1,1) \in \mathcal{S}_{S-}$ ⋆, as well as $(-1,1) \in \mathcal{S}_{S0}$. Hence this triplet contains precisely the BPS states that are responsible for the singularities and that become massless at u_1, u_2, u_3 on $\mathcal{C}_1$, as one may expect. Now we are going to show that this is the only $\mathbf{Z}_3$ triplet in $\mathcal{S}_{S0}$. Suppose that $(n_e,n_m) \in \mathcal{S}_{S0}$. If $n_m = 0$, this is either $(1,0)$ or $(n,0)$, $n \geq 2$. In the first case it is part of the triplet just discussed. If it were $(n,0)$ then it would be part of a triplet $(n,0)$, $(0,n)$, $(-n,n)$ in $\mathcal{S}_{S0}$, and $(0,n)$ would be an additional massless state at u_3, which we know does not exist. So suppose $n_m \neq 0$. With (n_e,n_m) also $(-n_m, n_e+n_m)$ and $(-n_e-n_m, n_e)$ must be in $\mathcal{S}_{S0}$. Now one of these three

⋆ We explain how to determine the correct signs below.

states will become massless on $\mathcal{C}^0$, which is the part of the curve $\mathcal{C}_1$ in $\mathcal{R}_{S0}$, if one of the three expressions

$$\frac{n_e}{n_m} \equiv r \ , \quad \frac{-n_m}{n_e+n_m} = -\frac{1}{1+r} \equiv \varphi_1(r) \ , \quad \frac{-n_e-n_m}{n_e} = -\left(1+\frac{1}{r}\right) \equiv \varphi_2(r) \tag{3.32}$$

is in the interval $[-1,0]$. It is easy to see that the functions $\varphi_1(r)$ and $\varphi_2(r)$ are such that either r or $\varphi_1(r)$ or $\varphi_2(r)$ is always in $[-1,0]$. Hence one or the other state of the triplet will always become massless somewhere on $\mathcal{C}_1$. This contradicts the singularity structure and the triplet cannot be in $\mathcal{S}_{S0}$, unless it is the above-mentioned triplet containing $(0,1)$. Thus

$$\begin{aligned} \mathcal{S}_{S0} &= \{\pm(0,1) \ , \ \pm(-1,1) \ , \ \pm(1,0)\} \\ \mathcal{S}_{S+} &= \{\pm(1,0) \ , \ \pm(-1,1) \ , \ \pm(2,-1)\} = M_{u_1}\mathcal{S}_{S0} \\ \mathcal{S}_{S-} &= \{\pm(0,1) \ , \ \pm(-1,0) \ , \ \pm(1,1)\} = M_{u_3}^{-1}\mathcal{S}_{S0} \ . \end{aligned} \tag{3.33}$$

3.5. Disintegrations

We will now show that all states of $\mathcal{S}_W$ can consistently decay into the states of the strong-coupling spectrum when crossing the curve of marginal stability. Suppose that the curve is crossed somewhere on $\mathcal{C}^0$, so that the available states are those of $\mathcal{S}_{S0}$. (The other two possibilities can be discussed in exactly the same way and do work out consistently, too).

First recall the flavour charges S of these states in the strong-coupling region $\mathcal{R}_{S0}$. Since the (anti)monopole $(0,\pm1)$ in $\mathcal{R}_{S0}$ can be obtained by continuous deformation from the semi-classical one when crossing the curve $\mathcal{C}_1$ at u_3, it also has $S=\pm1/2$. Similarly, the (anti)dyon $(\mp1,\pm1)$ can cross $\mathcal{C}_1$ at u_1 and hence has $S=\mp1/2$. To determine the charge S of $(1,0)\in\mathcal{S}_{S0}$, note that it is described as $\pm(-2,1)\in\mathcal{S}_{S+}$ and as $\mp(1,1)\in\mathcal{S}_{S-}$ where it is massless at u_2 and can be connected to the semi classical states. In both cases one obtains $S=\pm1/2$ in $\mathcal{S}_{S+}$ and $\mathcal{S}_{S-}$. However, as noted above, we do not know whether the state corresponding to $(1,0)$ in $\mathcal{S}_{S0}$ is, e.g. in $\mathcal{S}_{S+}$, $(2,-1)$ with $S=-1/2$ or rather $(-2,1)$ with $S=1/2$. Hence we only know that $(\pm1,0)$ has $|S|=1/2$ but we cannot at first sight determine the sign. We will see that *all* decay reactions work out consistently if we assume that $(\pm1,0)$ has $S=\pm1/2$, while for the opposite choice almost all decays would violate the conservation of the S charge. As we know that the decay reactions *must* take place, as was shown above using arguments involving

only the electric and magnetic quantum numbers, we see that we can determine in this way the sign of the flavor charge in all cases where an ambiguity remained. Below we list again the strong-coupling spectrum, now denoting the states as $(n_e, n_m)_S$ with the flavour charge quoted explicitly.

$$
\begin{aligned}
\mathcal{S}_{S0} &= \left\{(0,\pm1)_{\pm1/2}\ ,\ (\mp1,\pm1)_{\mp1/2}\ ,\ (\pm1,0)_{\pm1/2}\right\}\\
\mathcal{S}_{S+} &= \left\{(\mp1,0)_{\pm1/2}\ ,\ (\mp1,\pm1)_{\mp1/2}\ ,(\mp2,\pm1)_{\pm1/2}\right\}\\
\mathcal{S}_{S-} &= \left\{(0,\pm1)_{\pm1/2}\ ,\ (\pm1,0)_{\mp1/2}\ ,\ (\mp1,\mp1)_{\pm1/2}\right\}.
\end{aligned}
\tag{3.34}
$$

In this notation the weak-coupling spectrum contains the states $(\pm2,0)_0$, $(\pm1,0)_{\pm1}$, $(\pm1,0)_{\mp1}$, $(\pm2n,\pm1)_{\pm1/2}$ and $(\pm(2n+1),\pm1)_{\mp1/2}$, cf. (3.22). All decay reactions across $\mathcal{C}^0$ must preserve the total charges of n_e, n_m and S. One indeed has

$$
\begin{aligned}
\begin{pmatrix}\pm1\\0\end{pmatrix}_{\pm1} &\longleftrightarrow \begin{pmatrix}0\\\pm1\end{pmatrix}_{\pm1/2} + \begin{pmatrix}\pm1\\\mp1\end{pmatrix}_{\pm1/2}\\
\begin{pmatrix}\pm1\\0\end{pmatrix}_{\mp1} &\longleftrightarrow 3\times\begin{pmatrix}0\\\mp1\end{pmatrix}_{\mp1/2} + 3\times\begin{pmatrix}\mp1\\\pm1\end{pmatrix}_{\mp1/2} + 4\times\begin{pmatrix}\pm1\\0\end{pmatrix}_{\pm1/2}\\
\begin{pmatrix}\pm2\\0\end{pmatrix}_{0} &\longleftrightarrow 2\times\begin{pmatrix}0\\\mp1\end{pmatrix}_{\mp1/2} + 2\times\begin{pmatrix}\mp1\\\pm1\end{pmatrix}_{\mp1/2} + 4\times\begin{pmatrix}\pm1\\0\end{pmatrix}_{\pm1/2}\\
\begin{pmatrix}\pm2n\\\pm1\end{pmatrix}_{\pm1/2} &\longleftrightarrow (2n-1)\times\begin{pmatrix}0\\\mp1\end{pmatrix}_{\mp1/2} + 2n\times\begin{pmatrix}\mp1\\\pm1\end{pmatrix}_{\mp1/2} + 4n\times\begin{pmatrix}\pm1\\0\end{pmatrix}_{\pm1/2}\\
\begin{pmatrix}\pm(2n+1)\\\pm1\end{pmatrix}_{\mp1/2} &\longleftrightarrow (2n+2)\times\begin{pmatrix}0\\\mp1\end{pmatrix}_{\mp1/2} + (2n+3)\times\begin{pmatrix}\mp1\\\pm1\end{pmatrix}_{\mp1/2}\\
&\qquad + (4n+4)\times\begin{pmatrix}\pm1\\0\end{pmatrix}_{\pm1/2}
\end{aligned}
\tag{3.35}
$$

and remarkably enough all quantum numbers work out consistently.

It remains to check the mass balance, but this also works out in all cases, just as it did for $N_f = 0$ [1]. For example, for the first disintegration in (3.35) one has on the l.h.s. $m = \sqrt{2}|a^{(1)}|$ while the mass on the r.h.s. is (let r denote $a_D^{(1)}/a^{(1)} \in [-1,0]$ since one crosses the curve on

$\mathcal{C}^0$)

$$m = \sqrt{2}\left(|a_D^{(1)}| + |a_D^{(1)} + a^{(1)}|\right) = \sqrt{2}|a^{(1)}|\left(|r| + |1+r|\right) = \sqrt{2}|a^{(1)}|\left(-r+1+r\right) = \sqrt{2}|a^{(1)}| \tag{3.36}$$

as it should.

4. Two flavours of quarks

4.1. Global analytic structure, $\mathbf{Z}_2$ symmetry and BPS spectrum

As was argued in [6], we have for $N_f = 2$ two singularities on the Coulomb branch, where BPS particles in a two dimensional representation of $Spin(4) = SU(2) \times SU(2)$ become massless. Because of the $\mathbf{Z}_2$ symmetry, they must be located at two points related by this symmetry; we choose the scale of the theory so that the singular points are at $u = 1$ and $u = -1$. Suppose that the singularity at $u = 1$ is produced by a state (n_e, n_m) in the $(\mathbf{2}, \mathbf{1})$ of $SU(2) \times SU(2)$ (this fixes our convention for the order of the $SU(2)$ factors of $Spin(4)$). Thus, n_e is even and n_m is odd, see Section 2.2. The associated monodromy matrix is given by (2.4) with $d = 2$. According to the discussion in Section 2.2, the state becoming massless at $u = -1$ must then be $(n_e \pm n_m, n_m)$. Thus it lies in the $(\mathbf{1}, \mathbf{2})$ representation, in accordance with the fact that $\mathbf{Z}_2$ contains the ρ transformation. The general consistency condition for the monodromy group is

$$M_{(n_e,n_m),2}\, M_{(n_e+n_m,n_m),2} = M_\infty = M_{(n_e-n_m,n_m),2}\, M_{(n_e,n_m),2}, \tag{4.1}$$

and this implies $n_m = \pm 1$. The electric charge is left arbitrary, reflecting the possibility of "democracy" transformations, and we choose $n_e = 0$. This shows that, in this case again, the "minimal" choice of [6] is actually the only possible one.

Now, one may note that the monodromies are exactly the same as in the case of the $N_f = 0$ theory in the old conventions of [5] for a and the electric charge. This, together with the asymptotics (2.2), completely determines the solution to be

$$\begin{aligned} a^{(2)}(u) &= a(u) \\ a_D^{(2)}(u) &= \frac{1}{2} a_D(u) \ , \end{aligned} \tag{4.2}$$

where the functions $a(u)$ and $a_D(u)$ are defined by (2.6). It follows that $a_D^{(2)}/a^{(2)} = a_D/2a$

and that the curve of marginal stability $\mathcal{C}_2$ for $N_f = 2$ is the same as for $N_f = 0$ (see Fig. 2). Furthermore, the $\mathbf{Z}_2$ symmetry is implemented formally exactly in the same way as in the case $N_f = 0$ discussed in [1].

The analysis of the spectrum can thus be done essentially without modification. In the weak coupling spectrum we now have the W bosons $(2,0)$ which are singlets of $SU(2)\times SU(2)$ and the elementary quarks $(1,0)$ in the $(\mathbf{2},\mathbf{2})$ (which is nothing but the defining representation of $SO(4)$), in addition to the dyons $(n,1)$. The latter are in the $(\mathbf{2},\mathbf{1})$ or in the $(\mathbf{1},\mathbf{2})$ according to whether n is even or odd.

In the strong-coupling region, one must introduce two different descriptions of the same section p over E_2, one for $\Im m\, u > 0$ (region $\mathcal{S}_{S+}$) and one for $\Im m\, u < 0$ (region $\mathcal{S}_{S-}$). We have

$$p \equiv (n_e, n_m) \in \mathcal{S}_{S+} \Longleftrightarrow p \equiv (n_e, n_m + 2n_e) \in \mathcal{S}_{S-} \ . \tag{4.3}$$

Note that this transformation law is compatible with the semi-classical formula for the constraints on the representations, since $n_m + 3n_e = n_m + n_e$ mod 2. This is related to the fact that here the strong-coupling region is separated into only two pieces, as already explained in Section 3. One then shows that the states must come in $\mathbf{Z}_2$ pairs which are described, for example in $\mathcal{S}_{S+}$, as $\big\{ \pm (n_e, n_m), \pm(n_e + n_m, -2n_e - n_m)\big\}$. Note that the two $SU(2)$ factors of the flavour group are interchanged for the two members of a given pair, as it should be. Finally, using our by now standard argumentation, one can show that there is only one $\mathbf{Z}_2$ pair in the strong-coupling spectrum containing the monopole $\pm(0,1)$ and the dyon $\pm(\pm 1,1)$.

$$\begin{aligned} \mathcal{S}_{S+} &= \{\pm(1,-1)\ ,\ \pm(0,1)\} = M_1 \mathcal{S}_{S-} \\ \mathcal{S}_{S-} &= \{\pm(1,1)\ ,\ \pm(0,1)\} = M_1^{-1} \mathcal{S}_{S+}\ . \end{aligned} \tag{4.4}$$

4.2. Disintegrations

We first list the predicted decay reactions, which here are completely fixed looking only at the electric and magnetic quantum numbers. We perform our analysis in $\mathcal{S}_{S+}$.

$$\begin{aligned}
\pm\begin{pmatrix}1\\0\end{pmatrix} &\longleftrightarrow \pm\begin{pmatrix}1\\-1\end{pmatrix}\pm\begin{pmatrix}0\\1\end{pmatrix}\\
\pm\begin{pmatrix}2\\0\end{pmatrix} &\longleftrightarrow \pm 2\times\begin{pmatrix}1\\-1\end{pmatrix}\pm 2\times\begin{pmatrix}0\\1\end{pmatrix}\\
\pm\begin{pmatrix}n\\1\end{pmatrix} &\longleftrightarrow \pm n\times\begin{pmatrix}1\\-1\end{pmatrix}\pm (n+1)\times\begin{pmatrix}0\\1\end{pmatrix}\\
&.
\end{aligned} \tag{4.5}$$

One could now check that these reactions are compatible with mass conservation and spin. This is very similar to what was already done in [1]. Here we focus on the flavour quantum numbers. Recall that $\pm(1,-1)$ is in $(\mathbf{1},\mathbf{2})$ and that $(0,1)$ is in $(\mathbf{2},\mathbf{1})$. The decay of the quark is possible since the latter transforms in $(\mathbf{2},\mathbf{2})$ and we have $(\mathbf{1},\mathbf{2})\otimes(\mathbf{2},\mathbf{1})=(\mathbf{2},\mathbf{2})$. The decay of the W is also possible since $(\mathbf{2},\mathbf{2})\otimes(\mathbf{2},\mathbf{2})$ contains the trivial representation.

What about the dyons? First note that $\mathbf{2}^{\otimes n}$ contains only $SU(2)$ representations of integer spin when n is even, and of half-integer spin when n is odd. Thus $(\mathbf{2}^{\otimes(n+1)},\mathbf{2}^{\otimes n})$ will contain $(\mathbf{2},\mathbf{1})$ but not $(\mathbf{1},\mathbf{2})$ if n is even, and it will contain $(\mathbf{1},\mathbf{2})$ but not $(\mathbf{2},\mathbf{1})$ if n is odd. But this is exactly what we need for the disintegration of the dyons to be possible, since $\pm(n,1)$ is in $(\mathbf{2},\mathbf{1})$ or in $(\mathbf{1},\mathbf{2})$ when n is even or odd.

5. Three flavours of quarks

5.1. The structure of the singularities and the global analytic structure

From the analysis of [6], we know that we have two singularities in the moduli space, one of them due to a massless BPS state (n_e, n_m) which is a singlet of the flavour group $SU(4)$ (and thus $2n_e + n_m = 0 \bmod 4$), and the other due to a state $(\tilde{n}_e, \tilde{n}_m)$ in the defining representation $\mathbf{4}$ of $SU(4)$ ($2\tilde{n}_e + \tilde{n}_m = 1 \bmod 4$). As there is no symmetry acting on the Coulomb branch (except CP), the states (n_e, n_m) and $(\tilde{n}_e, \tilde{n}_m)$ are a priori unrelated. Moreover, at the expense of shifting u, one can suppose that $(\tilde{n}_e, \tilde{n}_m)$ is massless at $u = 0$ and (n_e, n_m) is massless at $u = 1/4$ (the scale is then $\Lambda_3^2 = 64$). The consistency condition for the monodromy group is

$$M_{(n_e,n_m),1}\, M_{(\tilde{n}_e+\tilde{n}_m,\tilde{n}_m),4} = M_\infty = M_{(\tilde{n}_e,\tilde{n}_m),4}\, M_{(n_e,n_m),1}. \tag{5.1}$$

This leads to an intricate system of algebraic equations, which have an invariance corresponding to the democracy transformations. We did not try to solve this system explicitly. Instead, we give an argument which does not require any computations and which fits the line of reasoning promoted in this work.

In the problem at hand, we should again have a curve $\mathcal{C}_3$ of marginal stability, diffeomorphic to a circle, passing through the singularities (but we do not check the validity of this assertion for a general structure of the singularities). The analytic structure will be similar to the other cases where only two singularities are present ($N_f = 0, 2$). Because of the monodromy at infinity, a_D/a will take all values in the real interval $[\tilde{n}_e/\tilde{n}_m, \tilde{n}_e/\tilde{n}_m + 1] = [r, r+1]$ along $\mathcal{C}_3$. Now, we know since the work in [10,15] that all the dyons $\pm(2n+1, 2)$ with magnetic charge two do belong to the semi-classical spectrum, and are singlets of $SU(4)$. Since there is always an integer n such that $(2n+1)/2 \in [r, r+1]$, we see that this state must become massless somewhere on the curve. Of course this state is (n_e, n_m). The choice of an odd integer n_e is then purely conventional, and we will take $(n_e, n_m) = \pm(-1, 2)$. Then one may return to (5.1) and show that necessarily $(\tilde{n}_e, \tilde{n}_m) = \pm(-1, 1)$. This conclude our analysis of the structure of the singularities, and shows that the choice proposed in [6] is again the only possible one.

The elliptic curve is in the case at hand [6]

$$y^2 = (x-u)(x^2 - x + u) \tag{5.2}$$

and $\mathrm{d}a_D^{(3)}/\mathrm{d}u$ and $\mathrm{d}a^{(3)}/\mathrm{d}u$ are given by the corresponding period integrals. With appropriate contours γ_i on the elliptic curve, $a_D^{(3)}$ and $a^{(3)}$ are then given by

$$\frac{\sqrt{2}}{8\pi}\oint_{\gamma_i}\mathrm{d}x\frac{2u-x}{\sqrt{(x-u)(x^2-x+u)}}\ . \tag{5.3}$$

It is straightforward to show [11] that $a_D^{(3)}$ and $a^{(3)}$ given by the integrals (5.3) satisfy the differential equation

$$\left[z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}-\frac{1}{4}\right]\begin{pmatrix}a_D^{(3)}\\ a^{(3)}\end{pmatrix}=0\ , \tag{5.4}$$

where z can be chosen to be $4u$ or equally $1-4u$. This is the hypergeometric differential equation with $a=b=-\frac{1}{2}$, and $c=0$. We observe that this is exactly the same equation as for $N_f=0$, except that there $z=\frac{u+1}{2}$. Hence, we immediately see that two independent solutions to (5.4) are given by $a_D(8u-1)$ and $a(8u-1)$ with a_D and a defined in (2.6).

It remains to determine the correct normalisations and linear combinations. From the required asymptotics (2.2) as $u\to\infty$ together with (2.7) one concludes $a^{(3)}(u)=\frac{\sqrt{2}}{4}a(8u-1)$ and

$$a_D^{(3)}(u)=\frac{\sqrt{2}}{16}\Big[a_D(8u-1)-\gamma a(8u-1)\Big]\ . \tag{5.5}$$

To determine the constant γ, we simply require to have a massless dyon $(-1,2)$ at $u=1/4$, that is $a(1/4)+2a_D(1/4)=0$. This leads to $\gamma=2$ and then

$$\begin{aligned}a_D^{(3)}(u)&=\frac{\sqrt{2}}{16}\Big[a_D(8u-1)-2\,a(8u-1)\Big]\\&=\frac{\sqrt{2}}{4}\left[i\Big(u-\frac{1}{4}\Big)F\Big(\frac{1}{2},\frac{1}{2},2;1-4u\Big)-\sqrt{u}F\Big(-\frac{1}{2},\frac{1}{2},1;\frac{1}{4u}\Big)\right]\\a^{(3)}(u)&=\frac{\sqrt{2}}{4}a\big(8u-1\big)=\sqrt{\frac{u}{2}}\,F\Big(-\frac{1}{2},\frac{1}{2},1;\frac{1}{4u}\Big)\ .\end{aligned} \tag{5.6}$$

The positions of the branch cuts are shown in Fig. 9.

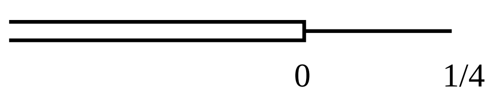

Fig. 9: The branch cuts of $a_D^{(3)}(u)$ and $a^{(3)}(u)$

The various asymptotics of $a_D^{(3)}$ and $a^{(3)}$ at the singularities are

$$\left.\begin{aligned} a_D^{(3)}(u) &\simeq \frac{i}{4\pi}\sqrt{2u}\,[\log u + 6\log 2 - 2 + i\pi] \\ a^{(3)}(u) &\simeq \frac{1}{2}\sqrt{2u} \end{aligned}\right\} \quad \text{as } u \to \infty$$

$$\left.\begin{aligned} a_D^{(3)}(u) &\simeq \frac{\sqrt{2}}{16\pi}\,[-4 + (4u-1)\log(4u-1) + (4u-1)(1 - 4\log 2 + i\pi)] \\ a^{(3)}(u) &\simeq \frac{\sqrt{2}}{8\pi}\,[4 - (4u-1)\log(4u-1) - (4u-1)(1 - 4\log 2)] \end{aligned}\right\} \quad \text{as } u \to \frac{1}{4}$$

$$\left.\begin{aligned} a_D^{(3)}(u) &\simeq \frac{\sqrt{2}}{16\pi} i(1+\epsilon)\,[-4 - 4u\log 4u + 4u(1 + 4\log 2)] - \frac{\sqrt{2}}{4}u \\ a^{(3)}(u) &\simeq \frac{\sqrt{2}}{8\pi} i\epsilon\,[4 + 4u\log 4u - 4u(1 + 4\log 2)] + \frac{\sqrt{2}}{2}u \end{aligned}\right\} \quad \text{as } u \to 0\ , \tag{5.7}$$

where again ϵ is the sign of $\Im m\ u$. It is then straightforward to determine the monodromy matrices for analytic continuation of $(a_D^{(3)}, a^{(3)})$ around these three singular points:

$$M_\infty = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\ , \ M_{1/4} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}\ , \ M_0 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\ , \ M_0' = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}\ , \tag{5.8}$$

where M_0 is to be used if the monodromy around $u = 0$ is computed with a basepoint in the lower half u-plane ($\epsilon = -1$) and M_0' if the basepoint is in the upper half u-plane ($\epsilon = +1$).

5.2. The curve of marginal stability

Let's now discuss the curve $\mathcal{C}$ of marginal stability. Again this is the set of all u such that $w^{(3)} \equiv a_D^{(3)}/a^{(3)}$ is real. Since

$$w^{(3)}(u) \equiv \frac{a_D^{(3)}}{a^{(3)}} = \frac{1}{4}\left[\frac{a_D(8u-1)}{a(8u-1)} - 2\right] \equiv \frac{1}{4}\,[w(8u-1) - 2]\ , \tag{5.9}$$

this curve is a rescaled and shifted copy of the curve for $N_f = 0$, which is almost an ellipse, see Fig. 10.

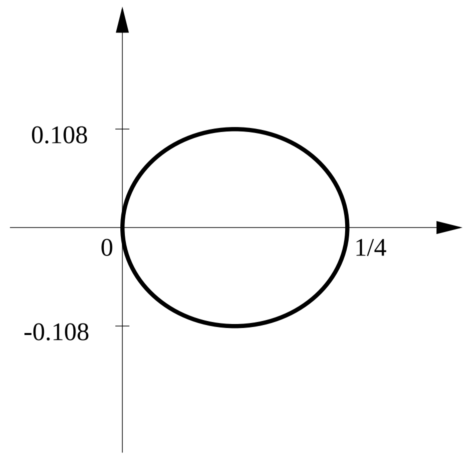

Fig. 10: The curve of marginal stability for $N_f = 3$

If we again call $\mathcal{C}^+$ and $\mathcal{C}^-$ the parts of the curve in the upper half and lower half u-plane, then

$$w^{(3)} \in [-1, -\frac{1}{2}] \text{ on } \mathcal{C}^+ \ , \quad w^{(3)} \in [-\frac{1}{2}, 0] \text{ on } \mathcal{C}^- \tag{5.10}$$

with $w^{(3)}(0+i\epsilon) = -1$, $w^{(3)}(\frac{1}{4}) = -\frac{1}{2}$ and $w^{(3)}(0-i\epsilon) = 0$, as follows immediately from the corresponding properties [1] of w we recalled in Sect. 2.1, but also from (5.7).

5.3. The weak and strong-coupling spectra

Next, we determine the weak and strong-coupling spectra $\mathcal{S}_W$ and $\mathcal{S}_S$, corresponding to the regions $\mathcal{R}_W$ and $\mathcal{R}_S$ outside and inside the curve $\mathcal{C}$. In all of $\mathcal{R}_W$, a BPS state can be described by a unique couple of integers (n_e, n_m). However, similarly to what happened for $N_f = 1$ or 2, in the strong-coupling region $\mathcal{R}_S$ one needs to introduce two descriptions (n_e, n_m) and $(\tilde{n}_e, \tilde{n}_m)$ for $\Im m\, u > 0$ (in $\mathcal{R}_{S+}$) and $\Im m\, u < 0$ (in $\mathcal{R}_{S-}$), for the *same* BPS state p. By exactly the same argument as for the other cases, this time one finds

$$p \equiv (n_e, n_m) \in \mathcal{S}_{S+} \quad \Longleftrightarrow \quad p \equiv (-n_e - n_m, 4n_e + 3n_m) \in \mathcal{S}_{S-} \ . \tag{5.11}$$

Now we are in a position to determine the strong-coupling spectrum. While the semi-classical monodromy M_∞ must be a symmetry of the weak-coupling spectrum, this need not and will not be the case for the strong-coupling spectrum. Also, here we do not have any broken global quantum symmetry $\mathbf{Z}_k$ at our disposal. But this is actually not needed. As is by now

familiar, a BPS state described by $(n_e,n_m)\in\mathcal{S}_{S+}$ will become massless somewhere on $\mathcal{C}^+$ if $\frac{n_e}{n_m}\equiv r\in[-1,-\frac{1}{2}]$. This same BPS state is described by $(\tilde n_e,\tilde n_m)\in\mathcal{S}_{S-}$, and it will be massless somewhere on $\mathcal{C}^-$ if

$$\frac{\tilde n_e}{\tilde n_m}=-\frac{n_e+n_m}{4n_e+3n_m}=-\frac{r+1}{4r+3}\equiv\varphi(r) \tag{5.12}$$

is such that $\varphi(r)\in[-\frac{1}{2},0]$. But it is easy to see that the function $\varphi(r)$ is such that either $r\in[-1,-\frac{1}{2}]$ (and then $\varphi(r)\notin(-\frac{1}{2},0)$) or $\varphi(r)\in[-\frac{1}{2},0]$ (for $r\notin(-1,-\frac{1}{2})$). Hence any BPS state in the strong-coupling spectrum becomes massless somewhere on the curve $\mathcal{C}$. But a massless BPS state always leads to a singularity. Since there are only two singularities, at $u=0$ and $u=\frac{1}{4}$, the only states in the strong-coupling spectrum are the two states (together with their anti-particle) responsible for the singularities, namely the magnetic monopole, described as $(0,1)\in\mathcal{S}_{S-}$ or $(1,-1)\in\mathcal{S}_{S+}$, and the dyon $(-1,2)\in\mathcal{S}_{S\pm}$:

$$\begin{aligned}\mathcal{S}_{S+}&=\{\pm(-1,2)\ ,\ \pm(1,-1)\}\\ \mathcal{S}_{S-}&=\{\pm(-1,2)\ ,\ \pm(0,1)\}\ .\end{aligned} \tag{5.13}$$

It remains to determine the weak-coupling spectrum $\mathcal{S}_W$. As already mentioned, the only symmetry here is the monodromy around $u=\infty$, and one must have

$$M_\infty\mathcal{S}_W=\mathcal{S}_W\ . \tag{5.14}$$

Since the states that become massless at $u=0$ and $u=\frac{1}{4}$ must be present in the weak and strong-coupling spectrum, we know that $(-1,2),(0,1)\in\mathcal{S}_W$. Applying $M_\infty^k\ ,\ k\in\mathbf{Z}$, to these states one obtains all dyons $(n,1)$ and $(2n{+}1,2)$, $n\in\mathbf{Z}$. The dyons $\pm(2n{+}1,2)$ obtained in this way are all singlets of $SU(4)$ since of course M_∞ does not change the flavour representations. One might ask whether it is possible to have dyons $\pm(2n+1,2)'$ which are not singlets of $SU(4)$ (note that for $|n_m|=1$ we are automatically in a spinor representation and thus a similar question would be irrelevant). But if this were the case, $\pm(-1,2)'$ would be massless at $u=1/4$ together with $\pm(-1,2)$, and thus $M_{1/4}$ would be changed, which is excluded. Of course, the elementary states, namely the quarks $(1,0)$ and W-bosons $(2,0)$ are also present in $\mathcal{S}_W$. Can one have dyons with magnetic charge $|n_m|\geq 3$? The answer is no. Suppose there were such

a dyon $(n_e, n_m) \in \mathcal{S}_W$ with $n_m \geq 3$. Then $\mathcal{S}_W$ would contain all dyons $(-)^k M_\infty^k (n_e, n_m) = (n_e - kn_m, n_m)$. There is always a $k \in \mathbf{Z}$ such that $\frac{n_e - kn_m}{n_m} = \frac{n_e}{n_m} - k \in [-1, 0]$. One would then conclude that $(n_e - kn_m, n_m)$ becomes massless somewhere on $\mathcal{C}$. But there are no massless states with $n_m \geq 3$. Hence we conclude that $\mathcal{S}_W$ cannot contain states with $n_m \geq 3$, and that

$$\mathcal{S}_W = \{\pm(1,0), \pm(2,0), \pm(n,1), \pm(2n+1,2),\ n \in \mathbf{Z}\} \ . \tag{5.15}$$

5.4. Disintegrations

As in the case $N_f = 2$, the predicted decay reactions are completely fixed when one takes into account the conservation of the electric and magnetic charge. We perform our analysis in $\mathcal{S}_{S-}$.

$$\begin{aligned}
\pm\begin{pmatrix}1\\0\end{pmatrix} &\longleftrightarrow \pm 2 \times \begin{pmatrix}0\\1\end{pmatrix} \pm \begin{pmatrix}1\\-2\end{pmatrix} \\
\pm\begin{pmatrix}2\\0\end{pmatrix} &\longleftrightarrow \pm 4 \times \begin{pmatrix}0\\1\end{pmatrix} \pm 2 \times \begin{pmatrix}1\\-2\end{pmatrix} \\
\pm\begin{pmatrix}n\\1\end{pmatrix} &\longleftrightarrow \pm(2n+1) \times \begin{pmatrix}0\\1\end{pmatrix} \pm n \times \begin{pmatrix}1\\-2\end{pmatrix} \\
\pm\begin{pmatrix}2n+1\\2\end{pmatrix} &\longleftrightarrow \pm 4(n+1) \times \begin{pmatrix}0\\1\end{pmatrix} \pm (2n+1) \times \begin{pmatrix}1\\-2\end{pmatrix}
\end{aligned} \quad . \tag{5.16}$$

We focus on the flavour quantum numbers, all other consistency conditions being satisfied, similarly to the previous cases. First we present some Clebsch-Gordan series for tensor products of the $\mathbf{4}$ of $SU(4)$ which actually will be enough to obtain the desired results. These are

$$\begin{aligned}
\mathbf{4}^{\otimes 2} &= \mathbf{6} \oplus \mathbf{10} \\
\mathbf{4}^{\otimes 3} &= \mathbf{20}_a \oplus 2 \times \mathbf{20}_b \oplus \overline{\mathbf{4}} \\
\mathbf{4}^{\otimes 4} &= \mathbf{1} \oplus 3 \times \mathbf{15} \oplus 2 \times \mathbf{20}_c \oplus \mathbf{35} \oplus 3 \times \mathbf{45}
\end{aligned} \quad . \tag{5.17}$$

We recall that the monopole $(0,1)$ is in the $\mathbf{4}$ and its anti-particle $(0,-1)$ is in the complex conjugate representation $\overline{\mathbf{4}}$, while $\pm(-1,2)$ is a singlet. Now it is clear that the decay reactions for the elementary quarks $(+1,0)$ are possible since $\mathbf{4} \otimes \mathbf{4}$ contains $\mathbf{6}$ which is the defining

representation of $SO(6)$ of which the quarks form a multiplet. Simply taking the complex conjugate of the Clebsch-Gordan series show that the corresponding reaction for the anti-particles is also possible (this is always the case, and we will thus only study the decays of $(+2,0)$, $(n,+1)$ and $(2n{+}1,+2)$ in the following). The reaction is possible for the W$^+$ because $\mathbf{1}$ is contained in $\mathbf{4}^{\otimes 4}$. Moreover, for n even, $\mathbf{4}^{\otimes(2n+1)} = (\mathbf{4}^{\otimes 4})^{\otimes(n/2)} \otimes \mathbf{4}$ contains $\mathbf{4}$ as $\mathbf{4}^{\otimes 4}$ contains $\mathbf{1}$, and thus the decay of the dyon of unit magnetic charge and even electric charge is consistent. The same is true when the electric charge is odd, because then $\mathbf{4}^{\otimes(2n+1)} = (\mathbf{4}^{\otimes 4})^{\otimes(n-1)/2} \otimes \mathbf{4}^{\otimes 3}$ and $\mathbf{4}^{\otimes 3}$ contains $\overline{\mathbf{4}}$. We end the discussion with the decay of the dyons of magnetic charge 2. Here, the desired result follows from the fact that $\mathbf{4}^{\otimes 4(n+1)}$ contains the trivial representation in its Clebsch-Gordan series.

6. Conclusions

In this paper, we have shown that the results of [1] for $N_f = 0$ can be extended to the general asymptotically free $N = 2$ supersymmetric Yang-Mills theories with gauge group $SU(2)$ when massless hypermultiplets are present. In all cases, we obtained the global structure of the $SL(2,\mathbf{Z})$ bundles, showed the existence and studied the properties of the curves of marginal stability, separating the strong from the so-called weak-coupling regions. For $N_f = 1$ and $N_f = 2$, we used the discrete global $\mathbf{Z}_3$ or $\mathbf{Z}_2$ symmetry on the Coulomb branch of the moduli space and showed that the states in the strong-coupling region must come in multiplets of the corresponding spontaneously broken discrete global symmetry $\mathbf{Z}_{4(4-N_f)}$. We saw for $N_f = 3$ that the existence of such a global symmetry was not crucial. Even for $N_f = 1$ and 2 one might not have used it explicitly: it is really the global structure of the moduli space which gives all the constraints we need. Of course this global structure is strongly constrained itself by the global symmetry when it exists.

In all cases we determined the strong and weak-coupling spectra of BPS states completely and rigorously. The strong-coupling spectra contain precisely only those states that are responsible for the singularities at finite points of the Coulomb branch. We showed that the discontinuities of the spectra across the curves of marginal stability lead to disintegrations that always are perfectly consistent with the conservation of mass, electric, magnetic and flavour charges. As a byproduct, we also showed that the electric and magnetic quantum

numbers of the massless states at the singularities proposed by Seiberg and Witten are the only possible ones.

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